Ejercicio de Orión

Determina si los conjuntos dados dotados con las operaciones de ⊕ y ⊗ son o no espacios vectoriales

(1.1) Sea H el conjunto de puntos dentro de un círculo en el plano:

$$H = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

① Conmutatividad en la suma

$$x^2 + y^2 \leq 1$$

$$= (x_1^2 + y_1^2 \leq 1, x_2^2 + y_2^2 \leq 1, \ldots, x_n^2 + y_n^2 \leq 1)$$

$$= (y_1^2 + x_1^2 \leq 1, y_2^2 + x_2^2 \leq 1, \ldots, y_n^2 + x_n^2 \leq 1)$$

$$= y^2 + x^2 \leq 1 \checkmark$$

② Asociatividad en la suma

$$x^2 + y^2 \leq 1 = \underline{x^2 + (y^2 - 1) \leq 0}$$

$$= (x_1^2 + (y_1^2 - 1) \leq 0, x_2^2 + (y_2^2 - 1) \leq 0, \ldots, x_n^2 + (y_n^2 - 1) \leq 0)$$

$$= ((x_1^2 + y_1^2) - 1 \leq 0, (x_2^2 + y_2^2) - 1 \leq 0, \ldots, (x_n^2 + y_n^2) - 1 \leq 0)$$

$= (x^2+y^2) - 1 \leq 0$

(3) Elemento neutro aditivo

Neutro aditivo $= 0$

$x^2+y^2 \leq 1 = x^2+0+y^2 \leq 1$

(4) Inverso aditivo

$x^2+y^2 \leq 1$

$x^2 - x^2 + y^2 \leq 1 - x^2$

$0 + y^2 \leq 1 - x^2$

$y^2 \leq 1 - x^2$

(5) Neutro multiplicativo

Neutro multiplicativo $= 1$

$x^2+y^2 \leq 1$

$1\cdot(x^2+y^2) \leq 1\cdot 1 = (1\cdot(x_1^2+y_1^2) \leq 1\cdot 1, \ldots, 1\cdot(x_n^2+y_n^2) \leq 1\cdot 1)$

$= (x_1^2+y_1^2 \leq 1, \ldots, x_n^2+y_n^2 \leq 1) = x^2+y^2 \leq 1$

⑥ Asociatividad del producto

Sean $\alpha, \beta \in \mathbb{R}$, $(x,y) \in \mathbb{R}^2$

$x^2 + y^2 \leq 1$

$\alpha\beta(x^2+y^2) \leq \alpha\beta$

$= (\alpha\beta(x_1^2+y_1^2) \leq \alpha\beta, \alpha\beta(x_2^2+y_2^2) \leq \alpha\beta, \dots, \alpha\beta(x_n^2+y_n^2) \leq \alpha\beta)$

$= (\alpha\beta x_1^2 + \alpha\beta y_1^2 \leq \alpha\beta, \alpha\beta x_2^2 + \alpha\beta y_2^2 \leq \alpha\beta, \dots, \alpha\beta x_n^2 + \alpha\beta y_n^2 \leq \alpha\beta)$

$= (\alpha(\beta x^2 + \beta y^2) \leq \alpha\beta, \alpha(\beta x_2^2 + \beta y_2^2 \leq \alpha\beta, \dots, \alpha(\beta x_n^2 + \beta y_n^2 \leq \alpha\beta)$

$= \alpha(\beta x^2 + \beta y^2) \leq \alpha\beta$

$= \alpha(\beta(x^2+y^2)) \leq \alpha\beta$ ✓

⑦ Distributividad

$\alpha, \beta \in \mathbb{R}$, $(x,y) \in \mathbb{R}^2$

$\alpha \cdot (x^2 + y^2) \leq \alpha$

$= (\alpha(x^2+y^2 \leq 1), \alpha(x_2^2+y_2^2 \leq 1), \dots, \alpha(x_n^2+y_n^2 \leq 1)$

$= (\alpha x_1^2 + y_1^2 \leq \alpha, \alpha x_2^2 + y_2^2 \leq \alpha, \dots, \alpha x_n^2 + \alpha y_n^2 \leq \alpha)$

$= \alpha x^2 + \alpha y^2 \leq \alpha$ ✓

⑧ Distributividad

Sean $\alpha, \beta \in \mathbb{R}$, $x, y \in \mathbb{R}^2$

$(\alpha + \beta)(x^2 + y^2 \leq 1)$

$= ((\alpha+\beta)(x_1^2 + y_1^2 \leq 1), \ldots, (\alpha+\beta)(x_n^2 + y_n^2 \leq 1))$

$= ((\alpha+\beta)(x_1^2) + (\alpha+\beta)(y_1^2) \leq \alpha+\beta, \ldots, (\alpha+\beta)(x_n^2) + (\alpha+\beta)(y_n^2) \leq \alpha\beta)$

$= (\alpha x_1^2 + \beta x_1^2 + \alpha y_1^2 + \beta y_1^2 \leq \alpha+\beta, \ldots, \alpha x_n^2 + \beta x_n^2 + \alpha y_n^2 + \beta y_n^2 \leq \alpha\beta)$

$= \alpha x^2 + \beta x^2 + \alpha y^2 + \beta y^2 \leq \alpha + \beta$ ✓

Por lo tanto $x^2 + y^2 \leq 1$ sí es espacio vectorial